

On some properties on bivariate Fibonacci and Lucas polynomials

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Abstract

In this paper we generalize to bivariate polynomials of Fibonacci and Lucas, properties obtained for Chebyshev polynomials. We prove that the coordinates of the bivariate polynomials over appropriate basis are families of integers satisfying remarkable recurrence relations.

1 Introduction

In [4], the authors established that Chebyshev polynomials of the first and second kind admit remarkable integer coordinates on specific basis. It turns out that this property can be extended to Jacobsthal polynomials [6, 7], Vieta polynomials [16, 10, 13, 14], and Morgan-Voyce polynomials [12, 2, 9, 11, 1, 15, 5] and Quasi-Morgan-Voyce polynomials [8], and more generally to bivariate polynomials associated to recurrence sequences of order two.

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by $(U_n) = (U_n(x, y))$ and $(V_n) = (V_n(x, y))$, are polynomials belonging to $\mathbb{Z}[x, y]$ and defined by

$$\begin{cases} U_0 = 0, U_1 = 1, \\ U_n = xU_{n-1} + yU_{n-2}, (n \geq 2) \end{cases} \quad \text{and} \quad \begin{cases} V_0 = 2, V_1 = x, \\ V_n = xV_{n-1} + yV_{n-2}, (n \geq 2) \end{cases}$$

It is established, see for example [3], that

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k, \quad (1)$$

$$V_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k. \quad (2)$$

Let \mathcal{E}_n be the \mathbb{Q} -vectorial space spanned by the free family $\mathcal{C}_n = (x^{n-2k} y^k)_k$, $0 \leq k \leq \lfloor n/2 \rfloor$. Thus the relations (1) and (2) appear as the decompositions of U_{n+1} and V_n over the canonical basis \mathcal{C}_n of \mathcal{E}_n .

Let us set

$$\begin{aligned}\mathcal{B}_U &= \mathcal{B}_{n,U} = (x^{n-k}U_{n+k+1})_{0 \leq k \leq n} \\ \mathcal{B}_V &= \mathcal{B}_{n,V} = (x^{n-k}V_{n+k})_{0 \leq k \leq n} \\ \mathcal{B}_U^* &= \mathcal{B}_{n,U}^* = (x^{n-k}U_{n+k})_{0 \leq k \leq n-1} \\ \mathcal{B}_V^* &= \mathcal{B}_{n,V}^* = (x^{n-k}V_{n+k-1})_{0 \leq k \leq n-1}.\end{aligned}$$

The goal of this paper is to prove that for $n \geq 1$, \mathcal{B}_U and \mathcal{B}_V (resp. \mathcal{B}_U^* and \mathcal{B}_V^*) are basis of E_{2n} (resp. E_{2n-1}) with respect to which, the polynomials U_{2n+1} and V_{2n} (resp. U_{2n} and V_{2n-1}) admit remarkable integer coordinates.

2 Main results

Theorem 1 *We have the following results*

1. $\mathcal{B}_{n,U}$ and $\mathcal{B}_{n,V}$ are basis of E_{2n} ,
2. $\mathcal{B}_{n,U}^*$ and $\mathcal{B}_{n,V}^*$ are basis of E_{2n-1} .

As U_{n+1} and V_n belong to E_n , the polynomials U_{2n+1} and V_{2n} are elements of E_{2n} with basis \mathcal{B}_U or \mathcal{B}_V . Similarly, U_{2n} and V_{2n-1} belong to E_{2n-1} with basis \mathcal{B}_U^* or \mathcal{B}_V^* .

Therefore, there are a priori 8 possible decompositions:

$$\begin{array}{ll} \begin{array}{l} U_{2n+1} \begin{array}{l} \nearrow \\ \searrow \end{array} \end{array} & \begin{array}{l} \text{over } \mathcal{B}_U \boxed{1} \rightarrow \text{trivial,} \\ \text{over } \mathcal{B}_V \boxed{2} \rightarrow \text{Th. C,} \\ \text{over } \mathcal{B}_U^* \boxed{5} \rightarrow \text{Th. A,} \\ \text{over } \mathcal{B}_V^* \boxed{6} \rightarrow \text{Th. D,} \end{array} & \begin{array}{l} V_{2n} \begin{array}{l} \nearrow \\ \searrow \end{array} \end{array} & \begin{array}{l} \text{over } \mathcal{B}_U \boxed{3} \rightarrow \text{simple,} \\ \text{over } \mathcal{B}_V \boxed{4} \rightarrow \text{trivial,} \\ \text{over } \mathcal{B}_U^* \boxed{7} \rightarrow \text{Th. E,} \\ \text{over } \mathcal{B}_V^* \boxed{8} \rightarrow \text{Th. B,} \end{array} \\ \begin{array}{l} U_{2n} \begin{array}{l} \nearrow \\ \searrow \end{array} \end{array} & & \begin{array}{l} V_{2n-1} \begin{array}{l} \nearrow \\ \searrow \end{array} \end{array} & \end{array}$$

where the cases 1 and 4 are obvious since $U_{2n+1} \in \mathcal{B}_U$ and $V_{2n} \in \mathcal{B}_V$.

The decomposition of V_{2n} in \mathcal{B}_U is simple: we have $V_{2n} = 2U_{2n+1} - xU_{2n}$.

The remaining cases are established by the five following results.

Theorem 2 (A). *Decomposition of U_{2n+1} on basis \mathcal{B}_V .*

For every integer $n \geq 0$, one has

$$2U_{2n+1} = \sum_{k=0}^n a_{n,k} x^{n-k} V_{n+k},$$

where

$$a_{n,k} = (-1)^{k+1} \binom{n}{k} + 2(-1)^{n-k} \sum_{j=0}^n (-1)^j \binom{j}{n-k}.$$

Moreover, $(c_{n,k})_{n,k \geq 0}$ is a family of integers satisfying the following recurrence relations

$$\begin{cases} a_{n,k} = a_{n-1,k} - a_{n-1,k-1} + 2\delta_{n,k}, & (n \geq 1, k \geq 1) \\ a_{n,0} = 1 & (n \geq 0) \\ a_{0,k} = \delta_{0,k} & (k \geq 0). \end{cases}$$

($\delta_{i,j}$ being the Kronecker symbol).

The recurrence relations allow obtaining the following table

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	0	1						
3	1	-1	1	1					
4	1	-2	2	0	1				
5	1	-3	4	-2	1	1			
6	1	-4	7	-6	3	0	1		
7	1	-5	11	-13	9	-3	1	1	
8	1	-6	16	-24	22	-12	4	0	1

from which it follows that

$$\begin{aligned} 2U_1 &= V_0, \\ 2U_3 &= xV_1 + V_2, \\ 2U_5 &= x^2V_2 + 0V_3 + V_4, \\ 2U_7 &= x^3V_3 - x^2V_4 + xV_5 + V_6. \end{aligned}$$

Theorem 3 (B). *Decomposition of U_{2n} on basis \mathcal{B}_U^* .*

For every integer $n \geq 1$, one has

$$U_{2n} = \sum_{k=0}^{n-1} b_{n,k} x^{n-k} U_{n+k}$$

where

$$b_{n,k} = (-1)^{n-k+1} \binom{n}{k}.$$

Moreover, $(b_{n,k})_{n,k \geq 0}$ is a family of integers satisfying recurrence relations

$$\begin{cases} b_{n,k} = -b_{n-1,k} + b_{n-1,k-1} & (n \geq 1, k \geq 1), \\ b_{n,0} = (-1)^{n+1} & (n \geq 0), \\ b_{0,k} = -\delta_{0,k} & (k \geq 0). \end{cases}$$

The latter recurrence relations allow obtaining the following table

$n \setminus k$	0	1	2	3	4	5
0	-1					
1	1	-1				
2	-1	2	-1			
3	1	-3	3	-1		
4	-1	4	-6	4	-1	
5	1	-5	10	-10	5	-1

from which it follows that

$$\begin{aligned} U_2 &= xU_1 \\ U_4 &= -x^2U_2 + 2xU_3 \\ U_6 &= x^3U_3 - 3x^2U_4 + 3xU_5 \\ U_8 &= -x^4U_4 + 4x^3U_5 - 6x^2U_6 + 4xU_7 \end{aligned}$$

Theorem 4 (C). *Decomposition of V_{2n-1} on basis \mathcal{B}_U^* .*

For every integer $n \geq 1$, one has

$$V_{2n-1} = \sum_{k=0}^{n-1} c_{n,k} x^{n-k} U_{n+k} \text{ with } c_{n,k} = 2(-1)^{n-k+1} \binom{n}{k} - \delta_{n-1,k}$$

Moreover, $(c_{n,k})_{n,k \geq 1}$ is a family of integers satisfying recurrence relations

$$\begin{aligned} c_{n,k} &= -c_{n-1,k} + c_{n-1,k-1} - \delta_{n,k+2}, \quad (n \geq 2, k \geq 1) \\ c_{n,0} &= 2(-1)^{n+1} - \delta_{n,1} \quad (n \geq 1) \\ c_{1,k} &= \begin{cases} 1 & \text{if } k=0 \\ -2 & \text{if } k=1 \\ 0 & \text{if } k \geq 2 \end{cases} \end{aligned}$$

The latter recurrence relations allow obtaining the following table

$n \setminus k$	0	1	2	3	4	5
1	1	-2				
2	-2	3	-2			
3	2	-6	5	-2		
4	-2	8	-12	7	-2	
5	2	-10	20	-20	9	-2

from which we get

$$\begin{cases} V_1 = xU_1 \\ V_3 = -2x^2U_2 + 3xU_3 \\ V_5 = 2x^3U_3 - 6x^2U_4 + 5xU_5 \\ V_7 = -2x^4U_4 + 8x^3U_5 - 12x^2U_6 + 7xU_7 \end{cases}$$

Theorem 5 (D). *Decomposition of V_{2n-1} on basis \mathcal{B}_V^* .*

For every integer $n \geq 1$, one has

$$2V_{2n-1} = \sum_{k=0}^{n-1} d_{n,k} x^{n-k} V_{n+k-1} \text{ and } d_{n,k} = (-1)^{n-k+1} \frac{n+k}{n} \binom{n}{k}$$

Moreover, $(d_{n,k})_{n \geq 1, k \geq 0}$ is a family of integers satisfying recurrence relations

$$\begin{aligned} d_{n,k} &= -d_{n-1,k} + d_{n-1,k-1}, \quad (n \geq 2, k \geq 1) \\ d_{n,0} &= (-1)^{n+1} \quad (n \geq 1) \\ d_{1,k} &= \begin{cases} 1 & \text{if } k=0 \\ -2 & \text{if } k=1 \\ 0 & \text{if } k \geq 2 \end{cases} \end{aligned}$$

where recurrence relations allow obtaining the following table

$n \setminus k$	0	1	2	3	4	5	6
1	1	-2					
2	-1	3	-2				
3	1	-4	5	-2			
4	-1	5	-9	7	-2		
5	1	-6	14	-16	9	-2	
6	-1	7	-20	30	-25	11	-2

from which we obtain

$$\begin{cases} 2V_1 = xV_0 \\ 2V_3 = -x^2V_1 + 3xV_2 \\ 2V_5 = x^3V_2 - 4x^2V_3 + 5xV_4 \\ 2V_7 = -x^4V_3 + 5x^3V_4 - 9x^2V_5 + 7xV_6 \end{cases}$$

Theorem 6 (E). *Decomposition of U_{2n} on basis \mathcal{B}_V^* .*

For every integer $n \geq 1$, one has

$$2U_{2n} = \sum_{k=0}^{n-1} e_{n,k} x^{n-k} V_{n+k-1} \text{ with } e_{n,k} = \frac{1}{2} (a_{n-1,k} + d_{n,k}) + \delta_{n,k}.$$

Moreover, $(c_{n,k})_{n,k \geq 0}$ is a family of integers satisfying recurrence relations

$$e_{n,k} = -e_{n-1,k} + e_{n-1,k-1} + a_{n-1,k} \quad (n \geq 2, k \geq 1),$$

with $e_{n,0} = (1 - (-1)^n)/2$ ($n \geq 1$), and $e_{1,k} = \delta_{0,k}$ ($k \geq 0$).

The latter recurrence relations allow obtaining the following table

$n \setminus k$	0	1	2	3	4	5
1	1					
2	0	2				
3	1	-2	3			
4	0	2	-4	4		
5	1	-4	8	-8	5	
6	0	2	-8	14	-12	6

from which, we have

$$\begin{cases} 2U_2 = xV_0 \\ 2U_4 = 0x^2V_0 + 2xV_2 \\ 2U_6 = x^3V_2 - 2x^2V_3 + 3xV_4 \\ 2U_8 = 0x^4V_3 + 2x^3V_4 - 4x^2V_5 + 4xV_6 \end{cases}$$

3 Proof of Theorems

Theorem 1 follows from the following lemma.

Lemma 7 $\det_{\mathcal{C}_{2n}}(\mathcal{B}_{n,U}) = \det_{\mathcal{C}_{2n-1}}(\mathcal{B}_{n,U}^*) = 1$ and $\det_{\mathcal{C}_{2n}}(\mathcal{B}_{n,V}) = \det_{\mathcal{C}_{2n-1}}(\mathcal{B}_{n,V}^*) = 2$.

Proof. Let us prove only the first equality as the proofs of the other ones are similar.

$$\begin{aligned}\det_{\mathcal{C}_{2n}}(\mathcal{B}_{n,U}) &= \det_{\mathcal{C}_{2n}}(W_0, W_1, \dots, W_n) \quad \text{where } W_k = x^{n-k}U_{n+k+1} \\ &= \det_{\mathcal{C}_{2n}}(W_0, W_1 - W_0, \dots, W_{n-1} - W_{n-2}, W_n - W_{n-1}),\end{aligned}$$

However,

$$W_j - W_{j-1} = x^{n-j}U_{n+j+1} - x^{n-j+1}U_{n+j} = x^{n-j}(U_{n+j+1} - xU_{n+j}) = x^{n-j}yU_{n+j-1}.$$

Thus,

$$\det_{\mathcal{C}_{2n}}(\mathcal{B}_{n,U}) = \det_{\mathcal{C}_{2n}}(x^n U_{n+1}, x^{n-1}yU_n, x^{n-2}yU_{n+1}, \dots, yU_{2n-1})$$

The "component" of $W_0 = x^n U_{n+1}$ over x^{2n} is equal to 1.

The "component" of $x^{n-j}yU_{n+j-1}$ over x^{2n} is equal to 0, so we have

$$\begin{aligned}\det_{\mathcal{C}_{2n}}(\mathcal{B}_{n,U}) &= \det_{\mathcal{C}_{2n-2}}(x^{n-1}U_n, x^{n-2}U_{n+1}, \dots, U_{2n-1}) \\ &= \det_{\mathcal{C}_{2n-2}}(x^{n-1-j}U_{n+j})_{0 \leq j \leq n-1} \\ &= \det_{\mathcal{C}_{2n-2}}(\mathcal{B}_{n-1,U}) \\ &= \det_{\mathcal{C}_{2n-4}}(\mathcal{B}_{n-2,U}) = \dots = \det_{\mathcal{C}_0}(\mathcal{B}_{0,U}) = 1\end{aligned}$$

□

Let E, A_m, B_m, C_m, D_m and E_m be the operators of $(\mathbb{Q}[x, y])^{\mathbb{N}}$ defined by

$$E((W_n)_n) = (W_{n+1})_n$$

$$\begin{aligned}A_m &= (x - E)^m + 2 \sum_{k=1}^m E^k (x - E)^{m-k}, \quad (m \geq 0), \\ B_m &= -(E - x)^m, \quad (m \geq 0), \\ C_m &= 2E^m + 2B_m - xE^{m-1}, \quad (m \geq 1), \\ D_m &= (E - x)^{m-1}(x - 2E), \quad (m \geq 1), \\ E_m &= \frac{1}{2}(xA_{m-1} + D_m + 2C_m), \quad (m \geq 0),\end{aligned}$$

where E is the forward shift operator given by $EW_n = W_{n+1}$.

Then, we have

$$\begin{aligned}A_m &= \sum_{k=0}^m a_{m,k} x^{m-k} E^k \quad \text{with} \quad a_{m,k} = (-1)^{k+1} \binom{m}{k} + 2(-1)^{m-k} \sum_{j=0}^m (-1)^j \binom{j}{m-k} \\ B_m &= \sum_{k=0}^m b_{m,k} x^{m-k} E^k \quad \text{with} \quad b_{m,k} = (-1)^{m-k+1} \binom{m}{k} \\ C_m &= \sum_{k=0}^{m-1} c_{m,k} x^{m-k} E^k \quad \text{with} \quad c_{m,k} = 2(-1)^{m-k+1} \binom{m}{k} - \delta_{m-1,k} \\ D_m &= \sum_{k=0}^m d_{m,k} x^{m-k} E^k \quad \text{with} \quad d_{m,k} = \frac{(-1)^{m-k+1} (m+k)}{m} \binom{m}{k} \\ E_m &= \sum_{k=0}^{m-1} e_{m,k} x^{m-k} E^k \quad \text{with} \quad e_{m,k} = \frac{1}{2}(a_{m-1,k} + d_{m,k}) + \delta_{m,k}\end{aligned}$$

With these notations, relations stated by Theorems *A*, *B*, *C*, *D* and *E* may be expressed by means of the following relations

$$\begin{aligned} \text{a. } \forall n &\in \mathbb{N} & A_n V_n &= 2U_{2n+1} \\ \text{b. } \forall n &\in \mathbb{N} & B_n U_n &= 0 \\ \text{c. } \forall n &\in \mathbb{N}^* & C_n U_n &= V_{2n-1} \\ \text{d. } \forall n &\in \mathbb{N}^* & D_n V_{n-1} &= 0 \\ \text{e. } \forall n &\in \mathbb{N}^* & E_n V_{n-1} &= 2U_n \end{aligned}$$

which are to be proven. For this, the following lemma will be useful for us.

Lemma 8 *For every integers n and m , we have*

1. $(x - E)^n U_m = (-y)^n U_{m-n}$ and $(x - E)^n V_m = (-y)^n V_{m-n}$ ($m \geq n \geq 0$)
2. $V_n = 2U_{n+1} - xU_n$ ($n \geq 0$)
3. $V_n = U_{n+1} + yU_{n-1}$ ($n \geq 1$)
4. $\sum_{k=1}^n (-y)^{n-k} V_{2k} = U_{2n+1} - (-y)^n$ ($n \geq 0$)

Proof.

1. We proceed by induction on n , observing that for $n = 1$, we have $(x - E)^n U_m = xU_m - U_{m+1} = -yU_{m-1}$ and $(x - E)^n V_m = -yU_{m-1}$, for $m \geq 1$.
2. For every integer $n \in \mathbb{N}$, let us put $S_n := 2U_{n+1} - xU_n$. We observe that $S_0 = 2$, $S_1 = x$ and $S_n = xS_{n-1} + yS_{n-2}$ for $n \geq 2$. Thus, for every $n \in \mathbb{N}$, $V_n = S_n = 2U_{n+1} - xU_n$.
3. For every integer $n \geq 1$, we have from the latter relation $V_n = U_{n+1} + (U_{n+1} - xU_n) = U_{n+1} + yU_{n-1}$.
4. For every integer $n \in \mathbb{N}$, put $T_n := U_{2n+1} - \sum_{k=1}^n (-y)^{n-k} V_{2k}$. The relation to be proven is equivalent to $T_n = (-y)^n$ ($n \geq 0$). Then, we remark that from relation 1. of this lemma, we have for every integer $n \geq 1$

$$T_n + yT_{n-1} = U_{2n+1} + yU_{2n-1} - V_{2n} = 0$$

$(T_n)_{n \geq 0}$ is then a geometric sequence with multiplier $(-y)$ and of first term $T_0 = 1$. It follows that for every integer $n \in \mathbb{N}$, $T_n = (-y)^n$.

□

Proof of relations a., b., c., d. and e.

a. For every integer $n \in \mathbb{N}$, we have

$$\begin{aligned} A_n V_n &= ((x - E)^n + 2 \sum_{k=1}^n E^k (x - E)^{n-k}) V_n \\ &= (-y)^n V_0 + 2 \sum_{k=1}^n (-y)^{n-k} V_{2k} \quad (\text{from 1 of lemma}) \\ &= 2U_{2n+1} \end{aligned}$$

b. For every integer $n \in \mathbb{N}$, we have

$$\begin{aligned} B_n U_n &= -(E - x)^n U_n \\ &= -(-y)^n U_0 \quad (\text{from 1 of lemma}) \\ &= 0 \end{aligned}$$

c. For every integer $n \in \mathbb{N}^*$, we have

$$\begin{aligned} C_n U_n &= (2E^n + 2B_n - xE^{n-1}) U_n \\ &= 2U_{2n} + 2B_n U_n - xU_{2n-1}, \end{aligned}$$

however $B_n U_n = 0$ (from a.), thus

$$\begin{aligned} C_n U_n &= 2U_{2n} - xU_{2n-1} \\ &= V_{2n-1} \end{aligned}$$

d. For every integer $n \in \mathbb{N}^*$, we have

$$\begin{aligned} D_n V_{n-1} &= (x - 2E)(E - x) V_{n-1} \\ &= (x - 2E) V_0 \\ &= xV_0 - 2V_1 \\ &= 0 \end{aligned}$$

e. For every integer $n \in \mathbb{N}^*$, we have

$$\begin{aligned} E_n V_{n-1} &= \left(\frac{1}{2}x A_{n-1} + \frac{1}{2}D_n + E^n\right) V_{n-1} \\ &= \frac{1}{2}x A_{n-1} V_{n-1} + \frac{1}{2}D_n V_{n-1} + V_{2n-1} \end{aligned}$$

But $A_{n-1} V_{n-1} = 2U_{2n-1}$ (from c.) and $D_n V_{n-1} = 0$ (from b.), It follows that

$$\begin{aligned} E_n V_{n-1} &= xU_{2n-1} + V_{2n-1} \\ &= 2U_{2n} \quad (\text{From 2 of the lemma}). \end{aligned}$$

□

Remark 9 Theorems A, B, C, D and E generalize results obtained for the Chebyshev polynomials [4], Indeed,

$$\begin{aligned} \frac{1}{2}V_n(2x, 1) &= T_n(x) \text{ is the Chebyshev polynomials of the first kind,} \\ U_{n+1}(2x, 1) &= U_n(x) \text{ is the Chebyshev polynomials of the second kind,} \end{aligned}$$

with

$$\begin{cases} T_n(x) = 2xT_{n-1} - T_{n-2} \\ T_0 = 1, T_1 = x \end{cases} \quad \text{and} \quad \begin{cases} U_n(x) = 2xU_{n-1} - U_{n-2} \\ U_0(x) = 1, U_1 = 2x \end{cases}$$

References

- [1] R. André-Jeannin, A generalization of Morgan-Voyce polynomials, Fibonacci Quart. 32 (1994), 228–231.
- [2] R. André-Jeannin, Differential Properties of a General Class of Polynomials, Fibonacci Quart. 33 (1995), no. 5, 453–458.
- [3] H. Belbachir and F. Bencherif, Linear recurrent sequences and powers of a square matrix. Integers 6 (2006), A12, 17pp.
- [4] H. Belbachir and F. Bencherif, On some properties of Chebyshev polynomials. Discuss. Math. Gen. Algebra Appl., 28 (2), (2008).
- [5] R. X. F. Chen, L. W. Shapiro, On sequences G_n satisfying $G_n = (d+2)G_{n-1} - G_{n-2}$, J. Integer Seq., Vol. 10 (2007), Art. 07.8.1.
- [6] G. B. Djordjević, Generalized Jacobsthal polynomials, Fibonacci Quart. 38, 239–243, (2000).
- [7] A. F. Horadam, Jacobsthal representation polynomials, Fibonacci Quart. 35 (1997), 137–148.
- [8] A. F. Horadam, Quasi Morgan-Voyce polynomials and Pell convolutions, Applications of Fibonacci numbers, Vol. 8(Rochester, NY, 1998), 179–193, Kluwer Acad. Publ., Dordrecht, 1999.

- [9] A. F. Horadam, Associated Legendre polynomials and Morgan-Voyce polynomials, *Notes Number Theory Discrete Math.* 5 (1999), no. 4, 125-134.
- [10] A. F. Horadam, Vieta polynomials, *Fibonacci Quart.* 40 (2002), 223–232.
- [11] J. Y. Lee, On the Morgan-Voyce polynomial generalization of the first kind, *Fibonacci Quart.* 40 (2002), 59–65.
- [12] A. M. Morgan-Voyce, *IRE Trans. Circuit Theory* 6, no. 3, (1959), 321–322.
- [13] N. Robbins, Vieta triangular array and a related family of polynomials. *Internat. J. Math. Math. Sci.* 14 (2), (1991), 239–244.
- [14] A. G. Shannon and A. F. Horadam, Some relationships among Vieta, Morgan-Voyce and Jacobsthal polynomials, *Application of Fibonacci numbers*, Vol. 8, Ed. F. Howard, Dordrecht: Kluwer, (1999), 307–323.
- [15] M. N. S. Swamy, Generalization of modified Morgan-Voyce polynomials, *Fibonacci Quart.* 38 (2002), 8–16.
- [16] F. Vieta. *Opera Mathematica: Ad Angulus Sectiones*. (Theorema VI). Paris, (1615). (repris de [13])